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FREE VIBRATION AND STABILITY OF THICK ELASTIC PLATES SUBJECTED TO IN-PLANE FORCES

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Abstract—Natural frequencies and buckling loads of a simply supported thick plate subjected to in-plane initial tensile and/or compressive forces are analysed. By using the method of power series expansion of displacement components, a set of fundamental dynamic equations of a two-dimensional higher-order plate theory is derived through Hamilton's principle. Several sets of truncated approximate theories are applied to solve the eigenvalue problems of a thick plate. In order to assure the accuracy of the present theory, convergence properties of the minimum natural frequency and the buckling load for the out-of-plane problem of a simply supported square plate are examined in detail. It is noticed that the present approximate theories can predict the frequencies and buckling loads of an extremely thick plate more accurately compared to other refined theories and classical plate theory.

NOMENCLATURE

a, b	length of rectangular plates in x^1- and x^2-directions
D, G	bending stiffness and shear modulus of plates
D_{00}, E_1	Lamé's constants
E, ν	Young's modulus and Poisson's ratio of plates
h	thickness of plates
M	order of approximate theory
$\overset{(n)}{N}{}^{\alpha\beta}, \overset{(n)}{Q}{}^{\beta}, \overset{(n)}{T}$	stress resultants of nth order defined by eqn (8)
$\overset{(0)}{N}{}_0^{\alpha\beta}$	in-plane initial forces
r, s	displacement mode numbers in x^1- and x^2-directions
$s^{\alpha\beta}, s^{\alpha 3}, s^{33}$	stress components
$s_0^{\alpha\beta}$	initial stresses distributed uniformly in thickness direction
t	time parameter
$\overset{(n)}{u}_{rs}, \overset{(n)}{v}_{rs}, \overset{(n)}{w}_{rs}$	generalized displacement components defined by eqn (23)
v_α, v_3	displacement components
$\overset{(n)}{v}_\alpha, \overset{(n)}{v}_3$	expanded displacement components defined by eqn (2)
x^α, x^3	Cartesian coordinates ($x^1 = a\xi$, $x^2 = b\eta$)
[G]	mass matrix or geometrical stiffness matrix
[I]	unit matrix
[K]	stiffness matrix
{U}	generalized displacement vector
$\delta^{\alpha\beta}$	Kronecker's delta
$\gamma_{\alpha\beta}, \gamma_{\alpha 3}, \gamma_{33}$	strain components
$\overset{(n)}{\gamma}_{\alpha\beta}, \overset{(n)}{\gamma}_{\alpha 3}, \overset{(n)}{\gamma}_{33}$	expanded strain components defined by eqn (3)
κ	ratio of in-plane forces of x^2- and x^1-directions
λ	eigenvalue
Λ	dimensionless buckling load or in-plane initial force in x^1-direction defined by eqn (24)
ω	natural frequency
Ω	dimensionless natural frequency defined by eqn (24)
ρ	mass density of plates.

Greek lower case subscripts (e.g. α, β) are assumed to range over the integers 1, 2.

1. INTRODUCTION

The natural frequencies and buckling loads of thick plates calculated by using the classical plate theory based on the well-known Kirchhoff–Love hypothesis are usually overpredicted. In order to analyse a thick plate it may be necessary to take into account the effects of rotary inertia and higher-order deformation such as transverse shear deformation and thickness change.

Mindlin (1951) derived a set of equations of motion by including the effects of rotary inertia under the same assumption of displacement distributions in the classical theory. In Mindlin's theory, the distributions of stress components are not specified in the thickness direction of a plate and, therefore, stress boundary conditions on the surfaces of a plate cannot be satisfied. Although the first-order effects of transverse shear deformation have been taken into consideration, the effects of thickness change are neglected. A set of equations of motion for a thick plate with initial stress has been derived through a variational procedure by Herrmann and Armenakas (1962). Certain inconsistencies were found in their formulation. Brunelle and Robertson (1974) derived the equations for initially stressed Mindlin plates by taking into account the effects of transverse shear and rotary inertia. The static buckling behavior of a simply supported thick plate with in-plane initial compressive and bending stresses was studied. Srinivas *et al.* (1970) developed an exact three-dimensional analysis for the vibration problem of simply supported homogeneous and laminated rectangular plates. Some numerical results for the natural frequencies and the thickness variations of stresses and displacements were compared with those from the thin plate and Mindlin's theories. Recently a continuum three-dimensional Ritz formulation has been presented for the vibration analysis of thick plates by Liew *et al.* (1993). Some frequency data for thick plates have been shown to examine the limitations of the classical plate theory and Mindlin plate theory. Based on an assumed displacement field that accounts for transverse shear and in-plane extensional deformation, Brunelle and Robertson (1976) derived plate equations to study the effects of an arbitrary state of initial stress in the vibration problem of thick plates. The equations were also used to study the static buckling behavior of a simply supported thick plate subjected to combined initial compressive stress and bending stress. Accounting for parabolic distribution of the transverse shear strains through the thickness of the plate, Reddy and Phan (1985) have developed a higher-order shear deformation theory to determine the natural frequencies and buckling loads of isotropic, orthotropic and laminated plates. The solutions of simply supported plates were compared with the exact solutions of three-dimensional elasticity theory, the first-order shear deformation theory and the classical theory. Doong (1987) derived the governing equation for a thick rectangular plate according to a higher-order deformation theory by using the same displacement field as the second-order approximate theory in the present paper. The buckling loads and natural frequencies of simply supported plates have been obtained and compared with the results of Brunelle and Robertson (1976) and Reddy and Phan (1985). These existing theories are of almost the same order and may be applied to the analysis of not so thick plates. Beyond the limits of applicability of the existing thick plate theories, more refined approximate theories should be applied to analyse an extremely thick plate. The refined theory may be required to introduce the effects of rotary inertia and transverse shear deformations and/or thickness changes. As an extension of the classical thin plate theory, the applicability and reliability of the two-dimensional higher-order theory have been clarified in detail through the numerical results of static boundary-value problems of an extremely thick plate [Matsunaga (1986), (1992)]. It can be said that two-dimensional higher-order plate theories are very useful for the static analysis of a thick plate as extended theories of the classical thin plate theory.

This paper presents the application of approximate equations of a two-dimensional higher-order theory for the analysis of vibration and stability problems of a thick elastic plate. On the basis of the power series expansions of displacement components, a fundamental set of dynamic equations of a two-dimensional higher-order plate theory is derived through Hamilton's principle. Several sets of truncated approximate equations of the present theory are used to solve the vibration and stability problems of a simply supported thick plate. Following the Navier solution procedure, the displacement components are expanded into Fourier series that satisfy the simply supported boundary conditions. The governing equations of motion can be expressed separately for any fixed values of displacement mode numbers of r and s. The natural frequency of a thick plate subjected to in-plane forces is obtained by solving the eigenvalue problem numerically and the buckling load is determined when the natural frequency is vanished. The convergence properties of the present numerical solutions are shown to be accurate for the natural frequencies and buckling loads with respect to the order of approximate theories. The present results obtained by various sets of approximate theories are considered to be accurate enough for extremely thick plates and can be regarded as the benchmark data of the problem. It is noticed that the two-dimensional higher-order plate theories in the present paper can predict the frequencies and buckling loads of an extremely thick plate more accurately compared to other refined theories and classical plate theories.

2. FUNDAMENTAL EQUATIONS OF KINEMATICS OF THICK PLATE

Introducing the Cartesian coordinates x^α ($\alpha = 1, 2$), x^3 on the middle plane of a plate of uniform thickness h, the dynamic displacement components in a plate are expressed as

$$v_\alpha \equiv v_\alpha(x^\alpha, x^3; t), \quad v_3 \equiv v_3(x^\alpha, x^3; t), \tag{1}$$

where t denotes time. The displacement components may be expanded into power series of the thickness coordinate x^3 as follows:

$$v_\alpha = \sum_{n=0}^{\infty} \overset{(n)}{v}_\alpha(x^3)^n, \quad v_3 = \sum_{n=0}^{\infty} \overset{(n)}{v}_3(x^3)^n, \tag{2}$$

where $n = 0, 1, 2, \ldots \infty$.

Based on this expression of the displacement components, a set of the linear fundamental equations of kinematics of a two-dimensional higher-order plate theory can be summarized in the following.

2.1. *Strain–displacement relations*

Strain components may also be expanded as follows:

$$\gamma_{\alpha\beta} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha\beta}(x^3)^n, \quad \gamma_{\alpha 3} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha 3}(x^3)^n, \quad \gamma_{33} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{33}(x^3)^n \tag{3}$$

and strain–displacement relations can be written as

$$\overset{(n)}{\gamma}_{\alpha\beta} = \tfrac{1}{2}(\overset{(n)}{v}_{\alpha,\beta} + \overset{(n)}{v}_{\beta,\alpha}), \quad \overset{(n)}{\gamma}_{\alpha 3} = \tfrac{1}{2}\{(n+1)\overset{(n+1)}{v}_\alpha + \overset{(n)}{v}_{3,\alpha}\}, \quad \overset{(n)}{\gamma}_{33} = (n+1)\overset{(n+1)}{v}_3, \tag{4}$$

where a comma indicates partial differentiation with respect to the coordinate subscripts that follow.

Introducing stress components $s^{\alpha\beta}$, $s^{\alpha 3}$ and s^{33}, Hamilton's principle is applied to derive the equations of dynamic equilibrium and natural boundary conditions of a plate. In order to treat vibration and stability problems of a plate subjected to in-plane stresses $s_0^{\alpha\beta}$ which distribute uniformly in the thickness direction, additional works due to these stresses which are assumed to remain unchanged during vibrating and/or buckling are taken into consideration. The principle for the present problems may be expressed for an arbitrary time interval t_1 to t_2 as follows:

$$\int_{t_1}^{t_2}\left[\int_V (s^{\alpha\beta}\delta\gamma_{\alpha\beta} + 2s^{\alpha 3}\delta\gamma_{\alpha 3} + s^{33}\delta\gamma_{33} - \rho\ddot{v}^\alpha\delta v_\alpha - \rho\ddot{v}^3\delta v_3)\,\mathrm{d}V \right.$$
$$\left. + \int_V s_0^{\alpha\beta}(\overset{(0)}{v}{}^\lambda_{,\alpha}\delta\overset{(0)}{v}_{\lambda,\beta} + \overset{(0)}{v}{}^3_{,\alpha}\delta\overset{(0)}{v}_{3,\beta})\,\mathrm{d}V - \int_S (s_*^\alpha\delta v_\alpha + s_*^3\delta v_3)\,\mathrm{d}S\right]\mathrm{d}t = 0, \tag{5}$$

where the overdot indicates partial differentiation with respect to time and ρ denotes the mass density; $\mathrm{d}V$, the volume element; $\mathrm{d}S$, the element of area of the external bounding surface; s_*^α and s_*^3, the prescribed components of the stress vector on the surface of a plate which are expressed in terms of the prescribed stress components as follows:

$$s_*^\alpha = \nu_\beta s_*^{\alpha\beta} + \nu_3 s_*^{\alpha 3}, \quad s_*^3 = \nu_\beta s_*^{\beta 3} + \nu_3 s_*^{33}, \tag{6}$$

where ν_β and ν_3 denote the components of the outward unit vector normal to the external bounding surface of the plate.

2.2. *Equations of motion*

By performing the variation as indicated in eqn (5), the equations of motion are obtained as follows:

$$\delta \overset{(n)}{v}_\beta : \overset{(n)}{N}{}^{\alpha\beta}_{,\alpha} - n \overset{(n-1)}{Q}{}^{\beta} + \overset{(n)}{p}{}^{\beta} = \rho \sum_{m=0}^{\infty} f(n+m+1) \ddot{\overset{(m)}{v}}{}^{\beta} \quad (\text{for } n \geqslant 1)$$

$$\delta \overset{(0)}{v}_\beta : \overset{(0)}{N}{}^{\alpha\beta}_{,\alpha} + (\overset{(0)}{N}{}^{\alpha\lambda}_0 \overset{(0)}{v}{}^{\beta}_{,\lambda})_{,\alpha} + \overset{(0)}{p}{}^{\beta} = \rho \sum_{m=0}^{\infty} f(m+1) \ddot{\overset{(m)}{v}}{}^{\beta}$$

$$\delta \overset{(n)}{v}_3 : \overset{(n)}{Q}{}^{\alpha}_{,\alpha} - n \overset{(n-1)}{T} + \overset{(n)}{p}{}^{3} = \rho \sum_{m=0}^{\infty} f(n+m+1) \ddot{\overset{(m)}{v}}{}^{3} \quad (\text{for } n \geqslant 1)$$

$$\delta \overset{(0)}{v}_3 : \overset{(0)}{Q}{}^{\alpha}_{,\alpha} + (\overset{(0)}{N}{}^{\alpha\beta}_0 \overset{(0)}{v}{}^{3}_{,\alpha})_{,\beta} + \overset{(0)}{p}{}^{3} = \rho \sum_{m=0}^{\infty} f(m+1) \ddot{\overset{(m)}{v}}{}^{3}, \tag{7}$$

where $n, m = 0, 1, 2, \ldots \infty$.

The stress resultants are defined as follows:

$$\overset{(0)}{N}{}^{\alpha\beta}_0 = h s_0^{\alpha\beta}, \quad \overset{(n)}{N}{}^{\alpha\beta} = \int_{-h/2}^{+h/2} s^{\alpha\beta} (x^3)^n \, \mathrm{d}x^3, \quad \overset{(n)}{Q}{}^{\alpha} = \int_{-h/2}^{+h/2} s^{\alpha 3} (x^3)^n \, \mathrm{d}x^3, \quad \overset{(n)}{T} = \int_{-h/2}^{+h/2} s^{33} (x^3)^n \, \mathrm{d}x^3. \tag{8}$$

Load terms measured per unit area of the middle plane are expressed as

$$\overset{(n)}{p}{}^{\beta} = [s_*^{\beta 3} (x^3)^n]_{-h/2}^{+h/2}, \quad \overset{(n)}{p}{}^{3} = [s_*^{33} (x^3)^n]_{-h/2}^{+h/2}, \tag{9}$$

where the stress components marked with an asterisk denote the prescribed quantities on the upper and lower surfaces of a plate and the following function is defined as

$$f(k) \equiv \int_{-h/2}^{+h/2} (x^3)^{k-1} \, \mathrm{d}x^3 = \frac{1}{k} \left(\frac{h}{2}\right)^k [1 - (-1)^k] \tag{10}$$

$$= \begin{cases} 0 & (k : \text{even}) \\ \dfrac{2}{k} \left(\dfrac{h}{2}\right)^k & (k : \text{odd}), \end{cases}$$

where k is an integer.

2.3. *Constitutive relations*

For elastic and isotropic materials, the constitutive relations can be written as

$$s^{\alpha\beta} = D_{00}\delta^{\alpha\lambda}\delta^{\beta\nu}\gamma_{\lambda\nu} + E_1\delta^{\alpha\beta}(\gamma_3^3 + \delta^{\lambda\nu}\gamma_{\lambda\nu})$$

$$s^{\alpha 3} = D_{00}\delta^{\alpha\lambda}\gamma_\lambda^3, \quad s^{33} = D_{00}\gamma^{33} + E_1(\gamma_3^3 + \delta^{\lambda\nu}\gamma_{\lambda\nu}), \tag{11}$$

where $\delta^{\alpha\beta}$ is Kronecker's delta and Lamé's constants D_{00} and E_1 are defined by using Young's modulus E and Poisson's ratio ν as follows:

$$D_{00} \equiv \frac{E}{1+\nu}, \quad E_1 \equiv \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{12}$$

2.4. *Boundary conditions*

The equations of the boundary conditions on the upper and lower surfaces are expressed as

$$s^{\alpha 3} = s_*^{\alpha 3}, \quad s^{33} = s_*^{33} \tag{13}$$

and along the boundaries on the middle plane as follows:

$$\overset{(n)}{v}_\alpha = \overset{(n)}{v}{}_\alpha^* \quad \text{or} \quad \nu_\beta \overset{(n)}{N}{}^{\alpha\beta} = \nu_\beta \overset{(n)}{N}{}_*^{\alpha\beta}$$

$$\overset{(n)}{v}_3 = \overset{(n)}{v}{}_3^* \quad \text{or} \quad \nu_\beta \overset{(n)}{Q}{}^{\beta} = \nu_\beta \overset{(n)}{Q}{}_*^{\beta}, \tag{14}$$

where $n = 0, 1, 2, \ldots \infty$ and the quantities marked with an asterisk denote quantities prescribed along the boundaries on the middle plane of a plate.

2.5. *Stress resultants in terms of the expanded displacement components*

Stress resultants can be derived from eqns (8) and (11) and eqns (3) and (4) in terms of the expanded displacement components.

$$\overset{(n)}{N}{}^{\alpha\beta} = \sum_{m=0}^{\infty}\left[\frac{D_{00}}{2}\delta^{\alpha\lambda}\delta^{\beta\nu}(\overset{(m)}{v}_{\lambda,\nu} + \overset{(m)}{v}_{\nu,\lambda}) + E_1\delta^{\alpha\beta}\{(m+1)\overset{(m+1)}{v_3} + \tfrac{1}{2}\delta^{\lambda\nu}(\overset{(m)}{v}_{\lambda,\nu} + \overset{(m)}{v}_{\nu,\lambda})\}\right] f(n+m+1)$$

$$\overset{(n)}{Q}{}^{\alpha} = \sum_{m=0}^{\infty}\left[\frac{D_{00}}{2}\delta^{\alpha\lambda}\{(m+1)\overset{(m+1)}{v_\lambda} + \overset{(m)}{v}_{3,\lambda}\}\right] f(n+m+1)$$

$$\overset{(n)}{T} = \sum_{m=0}^{\infty}\left[(D_{00}+E_1)(m+1)\overset{(m+1)}{v_3} + \frac{E_1}{2}\delta^{\lambda\nu}(\overset{(m)}{v}_{\lambda,\nu} + \overset{(m)}{v}_{\nu,\lambda})\right] f(n+m+1), \tag{15}$$

where $n, m = 0, 1, 2, \ldots \infty$.

2.6. *Equations of motion in terms of the expanded displacement components*

The equations of motion can be expressed in terms of the expanded displacement components by using eqn (15) as

$$\delta\overset{(n)}{v}_\beta\colon \sum_{m=0}^{\infty}\Bigg[\{[\tfrac{1}{2}(D_{00}\delta^{\alpha\lambda}\delta^{\beta\nu} + E_1\delta^{\alpha\beta}\delta^{\lambda\nu})(\overset{(m)}{v}_{\lambda,\nu} + \overset{(m)}{v}_{\nu,\lambda}) + E_1\delta^{\alpha\beta}(m+1)\overset{(m+1)}{v_3}]_{,\alpha}$$

$$-\rho\overset{(m)}{\ddot{v}}{}^{\beta}\} f(n+m+1) - \frac{n}{2}D_{00}\delta^{\beta\lambda}[(m+1)\overset{(m+1)}{v_\lambda} + \overset{(m)}{v}_{3,\lambda}] f(n+m)\Bigg] + \overset{(n)}{p}{}^{\beta} = 0 \quad (\text{for } n \geqslant 1)$$

$$\delta \overset{(0)}{v}_\beta: \sum_{m=0}^{\infty} \{[\tfrac{1}{2}(D_{00}\delta^{\alpha\lambda}\delta^{\beta\nu}+E_1\delta^{\alpha\beta}\delta^{\lambda\nu})(\overset{(m)}{v}_{\lambda,\nu}+\overset{(m)}{v}_{\nu,\lambda})+E_1\delta^{\alpha\beta}(m+1)\overset{(m+1)}{v_3}]_{,\alpha}$$

$$-\rho\overset{(m)}{\ddot{v}^\beta}\}\, f(m+1)+(\overset{(0)}{N_0^{\alpha\lambda}}\overset{(0)}{v^\beta_{,\lambda}})_{,\alpha}+\overset{(0)}{p^\beta}=0$$

$$\delta \overset{(n)}{v}_3: \sum_{m=0}^{\infty}\left[\left\{\frac{D_{00}}{2}\delta^{\alpha\lambda}[(m+1)\overset{(m+1)}{v_\lambda}+\overset{(m)}{v}_{3,\lambda}]_{,\alpha}-\rho\overset{(m)}{\ddot{v}^3}\right\}f(n+m+1)\right.$$

$$\left.-n\left[(D_{00}+E_1)(m+1)\overset{(m+1)}{v_3}+\frac{E_1}{2}\delta^{\alpha\lambda}(\overset{(m)}{v}_{\alpha,\lambda}+\overset{(m)}{v}_{\lambda,\alpha})\right]f(n+m)\right]+\overset{(n)}{p^3}=0 \quad (\text{for } n\geqslant 1)$$

$$\delta \overset{(0)}{v}_3: \sum_{m=0}^{\infty}\left\{\frac{D_{00}}{2}\delta^{\alpha\lambda}[(m+1)\overset{(m+1)}{v_\lambda}+\overset{(m)}{v}_{3,\lambda}]_{,\alpha}-\rho\overset{(m)}{\ddot{v}^3}\right\}f(m+1)+(\overset{(0)}{N_0^{\alpha\beta}}\overset{(0)}{v^3_{,\alpha}})_{,\beta}+\overset{(0)}{p^3}=0. \tag{16}$$

Within the range of linear problems, the governing equations can be divided into two types of in-plane and out-of-plane problems according to the symmetry or antisymmetry conditions of loads and/or displacements with respect to the middle plane of a plate.

2.7. *M*th *order approximate theory*

Since the fundamental equations mentioned above are complex, approximate theories of various orders, they may be considered for the present problem. A set of the following combination of Mth ($M \geqslant 1$) order approximate equations is proposed (Matsunaga, 1986, 1992). This combination of the selected terms of displacement components is suggested from the forms of shear strain components in eqn (4).

(I) *In-plane problem*

$$v_\beta = \sum_{m=0}^{M-1}\overset{(2m)}{v_\beta}(x^3)^{2m}, \quad v_3=\sum_{m=0}^{M-2}\overset{(2m+1)}{v_3}(x^3)^{2m+1}, \tag{17}$$

where $M \geqslant 2$ for v_3.

(II) *Out-of-plane problem*

$$v_\beta = \sum_{m=0}^{M-1}\overset{(2m+1)}{v_\beta}(x^3)^{2m+1}, \quad v_3=\sum_{m=0}^{M-1}\overset{(2m)}{v_3}(x^3)^{2m}, \tag{18}$$

where $m = 0, 1, 2, 3, \ldots$.

The total number of the unknown displacement components is $(3M-1)$ for in-plane problems and $3M$ for out-of-plane problems.

In the above cases of $M = 1$, an assumption of plane strains is inherently imposed. For out-of-plane problems, another set of the governing equations of the lowest-order approximate theory ($M = 1$) which reduces to the classical theory is derived with the use of an assumption of plane state of stresses.

3. FOURIER SERIES SOLUTION FOR SIMPLY SUPPORTED PLATE

In order to show the applicability and reliability of a two-dimensional higher-order plate theory for the analysis of vibration and stability problems of a thick elastic plate, a simply supported rectangular plate of plan-form dimensions a and b and thickness h is analysed. The boundary conditions (14) can be expressed on the x^1-constant edges,

$$\overset{(n)}{N}{}^{11} = 0, \quad \overset{(n)}{v}_2 = 0, \quad \overset{(n)}{v}_3 = 0 \tag{19}$$

and on the x^2-constant edges,

$$\overset{(n)}{v}_1 = 0, \quad \overset{(n)}{N}{}^{22} = 0, \quad \overset{(n)}{v}_3 = 0. \tag{20}$$

For free vibration and buckling problems, load terms are set as follows:

$$\overset{(n)}{p}{}^1 = \overset{(n)}{p}{}^2 = \overset{(n)}{p}{}^3 = 0. \tag{21}$$

Since a plate is in a state of uniform stresses, the in-plane forces are considered to be constant and the following combination of the in-plane forces is taken into consideration:

$$\overset{(0)}{N}{}_0^{22} = \kappa \overset{(0)}{N}{}_0^{11}, \quad \overset{(0)}{N}{}_0^{12} = \overset{(0)}{N}{}_0^{21} = 0, \tag{22}$$

where κ denotes the ratio of in-plane forces of x^2- and x^1-directions.

Following the Navier solution procedure, displacement components that satisfy the equations of boundary conditions (19) and (20) may be expressed as

$$\overset{(n)}{v}_1 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{u}_{rs} \cos r\pi\xi \sin s\pi\eta \cdot e^{i\omega t}, \quad \overset{(n)}{v}_2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{v}_{rs} \sin r\pi\xi \cos s\pi\eta \cdot e^{i\omega t}$$

$$\overset{(n)}{v}_3 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{w}_{rs} \sin r\pi\xi \sin s\pi\eta \cdot e^{i\omega t}, \tag{23}$$

where $r, s = 1, 2, 3, \ldots \infty$, ω denotes the natural frequency and i the imaginary unit.

The equations of motion are rewritten in terms of the generalized displacement components $\overset{(n)}{u}_{rs}$, $\overset{(n)}{v}_{rs}$ and $\overset{(n)}{w}_{rs}$. The dimensionless natural frequency and buckling load or in-plane initial force in the x^1-direction for vibration problems are expressed as follows:

$$\Omega = \omega h \sqrt{\rho/G}, \quad \Lambda = Eh \overset{(0)}{N}{}_0^{11} b^2/\pi^2 D \tag{24}$$

where the bending stiffness and shear modulus of plates are defined as

$$D = Eh^3/12(1-\nu^2), \quad G = E/2(1+\nu). \tag{25}$$

4. EIGENVALUE PROBLEM FOR NATURAL FREQUENCY AND BUCKLING LOAD OF THICK PLATE

The equations of motion can be rewritten by collecting the coefficients for the generalized displacements of any fixed values r and s. The generalized displacement vector $\{\mathbf{U}\}$ is expressed for in-plane problems as

$$\{\mathbf{U}\}^{\mathrm{T}} = \{\overset{(0)}{u}_{rs}, \ldots, \overset{(2M-2)}{u}_{rs}; \; \overset{(0)}{v}_{rs}, \ldots, \overset{(2M-2)}{v}_{rs}; \; \overset{(1)}{w}_{rs}, \ldots, \overset{(2M-3)}{w}_{rs}\} \tag{26}$$

and for out-of-plane problems as

$$\{\mathbf{U}\}^{\mathrm{T}} = \{\overset{(1)}{u}_{rs}, \ldots, \overset{(2M-1)}{u}_{rs}; \; \overset{(1)}{v}_{rs}, \ldots, \overset{(2M-1)}{v}_{rs}; \; \overset{(0)}{w}_{rs}, \ldots, \overset{(2M-2)}{w}_{rs}\}. \tag{27}$$

Eigenvalue problems to determine the natural frequency and buckling load are generalized as follows:

$$([\mathbf{K}]-\lambda[\mathbf{G}])\{\mathbf{U}\} = 0, \tag{28}$$

where matrix [K] denotes the stiffness matrix which may contain the terms of the in-plane initial forces and matrix [G] refers to the mass matrix in the case of the vibration problem and to the geometric-stiffness matrix due to the in-plane forces in the case of the stability problem. The parameter λ refers to the corresponding frequency (Ω^2) or buckling load ($-\Lambda$).

In order to analyse the eigenvalue problems, eqn (28) may be rewritten as follows:

$$\left([\mathbf{K}]^{-1}[\mathbf{G}]-\frac{1}{\lambda}[\mathbf{I}]\right)\{\mathbf{U}\} = 0, \quad \rightarrow \quad \det\left([\mathbf{K}]^{-1}[\mathbf{G}]-\frac{1}{\lambda}[\mathbf{I}]\right) = 0, \tag{29}$$

where matrix [I] denotes the unit matrix. The matrix $[\mathbf{K}]^{-1}[\mathbf{G}]$ is called the dynamic matrix in the vibration problem and the stability matrix in the stability problem.

The power method is used to obtain the numerical solution of the eigenvalue problems. Although all the eigenvalues and eigenvectors can be computed by this method, the dominant eigenvalue which corresponds to the minimum natural frequency and/or buckling load is much concerned.

5. NUMERICAL EXAMPLES AND RESULTS

5.1. *Numerical examples*

A thick elastic rectangular plate with simply supported edges is analysed for six numerical examples with the thickness parameter

$$a/h = 1, \quad 2, \quad 4, \quad 5, \quad 10, \quad 20. \tag{30}$$

Poisson's ratio is fixed to be $\nu = 0.3$. All the numerical results are shown in the dimensionless quantities. Since the fundamental equations in the present problem are separated into two sets of equations of in-plane and out-of-plane problems, numerical results only for the out-of-plane problem are shown in the present examples.

In order to verify the accuracy of the present results, convergence properties of the numerical solutions according to the order of approximate theories are examined in detail. It is noticed that the proper order of present approximate theories may be estimated according to the level of thickness parameter of the plate. Although the present sets of approximate theories of any order can easily be applied to a moderately thick plate, higher orders of the expanded two-dimensional theories may be necessary to obtain reasonably accurate solutions for an extremely thick plate.

5.2. *Natural frequencies of a square plate without in-plane forces*

The minimum natural frequencies for the first three displacement modes of a square plate without in-plane forces are shown in Table 1. Convergence properties of the minimum natural frequency due to the Mth order of approximate theories have been examined and are shown for the thickness parameter $a/h = 2$ and 10. It is noticed that the present results are converged accurately enough within the present order of approximate theories.

In Table 1, for moderately thick plates with the thickness parameter $a/h = 10$ and 20, the results are compared with the values obtained by the classical plate theory. Since the classical plate theory overestimates the natural frequencies, higher-order effects of the present theories should be taken into account to obtain more accurate solutions for the problem. The results are compared with the exact values of the three-dimensional elasticity theory (Srinivas *et al.*, 1970) for the specific value of the thickness parameter $a/h = 10$. For extremely thick plates with the thickness parameter $a/h = 2$ and 5, the results are also compared with the recent results by a continuum three-dimensional Ritz formulation (Liew *et al.*, 1993).

Table 1. Minimum natural frequency Ω for the first three modes (r, s) of square plate $(a/b = 1)$

	r	1			2		3
a/h	s	1	2	3	2	3	3
1	$M = 5$	3.7419	6.2943	9.1017	8.0966	10.4266	12.3156
2	$M = 1$	1.5597	2.9180	4.4655	3.9089	5.2010	6.2509
	2	1.5185	2.8222	4.3138	3.7758	5.0278	6.0533
	3	1.5158	2.8067	4.2661	3.7420	4.9577	5.9437
	4	1.5158	2.8066	4.2658	3.7419	4.9571	5.9419
	5	1.5158	2.8066	4.2658	3.7419	4.9571	5.9419
	LHL	1.5158	2.8067	—	3.7419	—	—
4	$M = 5$	0.5066	1.0692	1.7744	1.5158	2.1219	2.6251
5	$M = 5$	0.3421	0.7511	1.2881	1.0889	1.5589	1.9557
	LHL	0.3421	0.7511	—	1.0889	—	—
10	CPT	0.0955	0.2360	0.4629	0.3732	0.5951	0.8090
	$M = 1$	0.0934	0.2241	0.4220	0.3454	0.5312	0.7008
	2	0.0932	0.2226	0.4172	0.3421	0.5240	0.6892
	3	0.0932	0.2226	0.4171	0.3421	0.5240	0.6892
	4	0.0932	0.2226	0.4171	0.3421	0.5239	0.6889
	5	0.0932	0.2226	0.4171	0.3421	0.5239	0.6889
	Exact	0.0932	0.2226	0.4171	0.3421	0.5239	0.6889
20	CPT	0.0240	0.0599	0.1192	0.0955	0.1545	0.2128
	$M = 5$	0.0239	0.0589	0.1155	0.0932	0.1485	0.2018

LHL: Liew *et al.* (1993); CPT: classical plate theory (including the rotary inertia); Exact: Srinivas *et al.* (1970).

Table 2. Buckling load Λ for various values of a/b and a/h when $\kappa = 0$ and $r = s = 1$

a/h		a/b						
		0.2	0.4	0.8	1.0	1.2	1.4	3.0†
10	A	26.050	8.068	3.966	3.729	3.798	3.737‡	2.359
	B	15.658	6.985	3.920	3.787	3.946	4.288	3.787
	C	26.093	8.080	3.969	3.730	3.797	3.751‡	2.384
	D	26.213	8.124	4.003	3.771	3.849	3.836‡	2.580
100	A	27.030	8.406	4.200	3.997	4.131	3.974	3.974
	B	26.843	8.393	4.200	3.998	4.132	4.468	3.998
	C	27.030	8.407	4.199	3.997	4.131	3.974	3.974
	D	27.032	8.407	4.200	3.998	4.131	4.466	3.978
	E	27.040	8.410	4.202	4.000	4.134	4.470	4.000

A: Brunelle and Robertson (1976); B: Reddy and Phan (1985); C: Doong (1987); D: Present sol. $(M = 5)$; E: classical plate theory (CPT).

† Minimum occurs at $r = 3$, $s = 1$.

‡ Minimum occurs at $r = 2$, $s = 1$.

The present results obtained by $M = 5$ with sufficient numerical accuracy can be regarded as the benchmark data of natural frequencies of a square thick plate. For an extremely thick plate, reasonably accurate natural frequencies are obtained by $M = 2$–5 according to the level of the thickness parameter.

5.3. *Buckling loads for* $\kappa = 0$

The buckling loads for the out-of-plane problem of a thick plate subjected to in-plane compressive force only in the x^1-direction ($\kappa = 0$) have been obtained. In the following tables, absolute values of the buckling loads for $s = 1$ are shown.

In Table 2, the critical buckling loads of a rectangular plate are compared with those of other analyses and the classical plate theory for the specific cases of the thickness parameter and the aspect ratio a/b. The present results are obtained by $M = 5$ which is considered to be accurate enough for the thickness parameters.

In Table 3, the buckling loads of a square plate for several displacement modes are shown. Convergence properties of the buckling load for the first displacement mode $r = 1$ due to the Mth order of approximate theories are also shown to be very satisfactory. The buckling loads for a higher displacement mode $r = 500$ are considered to be constant with

Table 3. Buckling load Λ for various values of M and a/h when $a/b = 1$ and $r = 1, 50, 100, 200, 500$; $s = 1$

	M ($r = 1$)					r ($M = 5$)			
a/h	1	2	3	4	5	50	100	200	500
1	0.7018	0.5580	0.5255	0.5007	0.4903	0.0719	0.0707	0.0704	0.0703
2	1.8391	1.5910	1.5805	1.5681	1.5620	0.2999	0.2875	0.2829	0.2814
4	3.0918	2.9014	2.9000	2.8974	2.8961	1.3303	1.1994	1.1501	1.1291
5	3.3670	3.2196	3.2189	3.2176	3.2170	2.2061	1.9184	1.8147	1.7683
X	3.6572	3.5687	3.5684	3.5681	3.5679	5.0341	4.0614	3.7206	3.5679
10	3.8204	3.7714	3.7713	3.7712	3.7712	11.9824	8.8221	7.6729	7.1879
20	3.9535	3.9403	3.9403	3.9403	3.9403	76.7487	47.9194	35.2861	29.9828

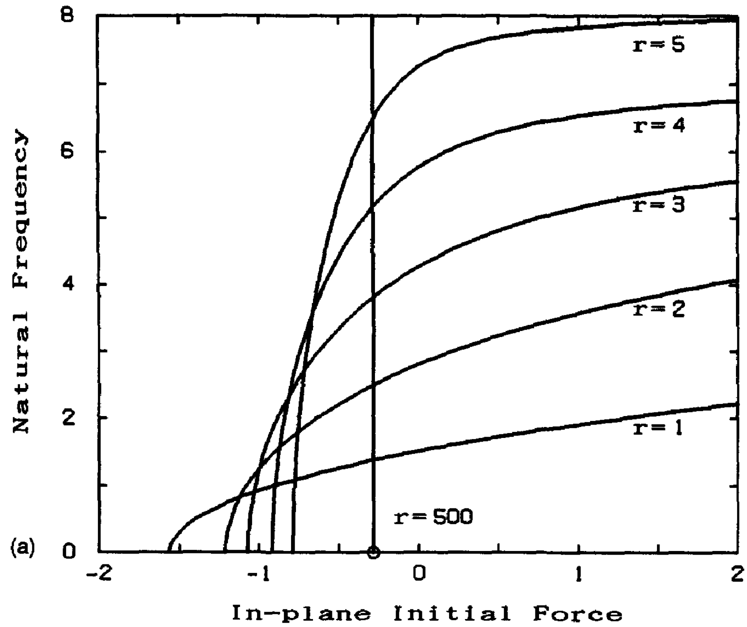
Minimum occurs at $r = 500$, $s = 1$ for $a/h \leqslant X$. $X = 7.08$ at which the same buckling load is obtained for $r = 1$ and 500.

Table 4. Buckling load Λ for various values of a/b and a/h when $r = 1, 2, 3, 500$; $s = 1$

		a/h					
a/b	r	1	2	4	5	10	20
3.00	1	0.1805	1.0687	3.7419	4.9606	8.5173	10.3276
	2	0.0534	0.3328	1.3042	1.7880	3.3616	4.2729
	3	0.0295	0.1891	0.8511	1.2173	2.5799	3.5190
	500	0.0078	0.0313	0.1255	0.1965	0.7987	3.3315
2.00	1	0.2577	1.2114	3.1837	3.8765	5.4258	6.0217
	2	0.0880	0.4903	1.5620	2.0134	3.2170	3.7712
	3	0.0534	0.3328	1.3042	1.7880	3.3616	4.2729
	500	0.0176	0.0703	0.2823	0.4421	1.7970	7.4958
1.00	1	0.4903	1.5620	2.8961	3.2170	3.7712	3.9403
	2	0.2577	1.2114	3.1837	3.8765	5.4258	6.0217
	3	0.1805	1.0687	3.7419	4.9606	8.5173	10.3276
	500	0.0703	0.2814	1.1291	1.7683	7.1879	29.9828
0.50	1	1.2114	3.1837	5.0498	5.4258	6.0217	6.1913
	2	0.9184	4.0367	9.9446	11.8864	15.9976	17.4986
	3	0.6868	3.9881	13.5084	17.7318	29.6724	35.5343
	500	0.2811	1.1255	4.5165	7.0731	28.7516	119.9308
0.33	1	2.4327	6.0951	9.3262	9.9510	10.9242	11.1975
	2	2.0569	8.9129	21.6671	25.8012	34.4584	37.5872
	3	1.5612	9.0290	30.3839	39.8061	66.2792	79.1851
	500	0.6454	2.5838	10.3684	16.2376	66.0045	275.3232
0.25	1	4.0367	9.9446	15.0283	15.9976	17.4986	17.9182
	2	3.5541	15.3266	37.0925	44.1143	58.7658	64.0431
	3	2.7114	15.6594	52.5819	68.8436	114.4384	136.6151
	500	1.1246	4.5020	18.0659	28.2924	115.0062	479.7230

respect to the variation of displacement mode r. The present results obtained by $M = 5$ with sufficient numerical accuracy can be regarded as the benchmark data of buckling loads of thick plates. It is noted that the lowest displacement mode gives the critical buckling load for thin plates. However, an interesting feature is that the critical buckling loads for thick plates occur at higher displacement modes. For this feature, a limit point of the thickness parameter at about $a/h = 7.08$ may appear in the present examples of a square plate. At this point the same value of critical buckling loads is obtained for the first displacement mode $r = 1$ and for a higher displacement mode $r = 500$.

In Table 4, the buckling loads of a rectangular plate for the first three displacement modes and for a higher displacement mode $r = 500$ are shown for several values of the aspect ratio. The results are obtained by $M = 5$ with sufficient numerical accuracy. It is noticed that critical buckling loads appear at higher displacement modes for the specific values of the thickness parameter and aspect ratio.



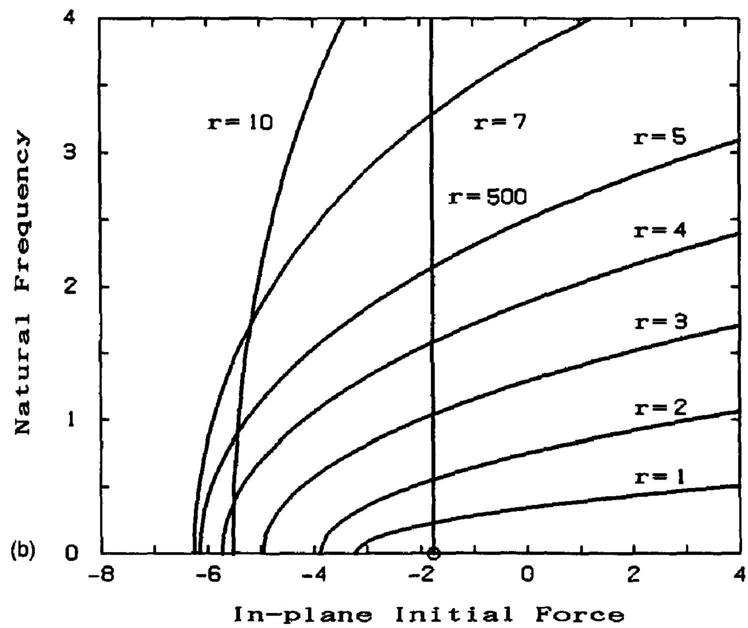


Fig. 1. Minimum natural frequency Ω versus in-plane initial force Λ for various values of displacement mode r: (a) $a/h = 2$; (b) $a/h = 5$.

5.4. *Natural frequencies of a square plate subjected to in-plane initial forces* ($\kappa = 0$)

The minimum natural frequencies of a square thick plate subjected to in-plane forces in the x^1-direction are plotted to the in-plane initial forces in Fig. 1. Frequency curves for the thickness parameters $a/h = 2$ and 5 are plotted for several displacement modes of r with $s = 1$. When the natural frequencies go to zero, the in-plane initial forces reduce to the buckling loads of the plate. The open circle on the horizontal axis shows the critical buckling load for a higher displacement mode $r = 500$. Numerical values of the buckling loads for the displacement modes of $r = 1$–3 and $r = 500$ are shown in Table 4.

It can be seen that frequency curves for higher displacement modes are more affected than those for lower displacement modes by increased compressive forces in the cases of thick plates. From Fig. 1 it can also be seen that thick plates will buckle in a higher mode $r = 500$. Frequency curves for lower displacement modes than the mode in which critical buckling occurs will cross above the curve for the critical buckling mode prior to buckling.

At the in-plane initial forces in the neighborhood of the critical buckling loads, lower natural frequencies can be found for higher displacement modes but not for lower ones.

6. DISCUSSIONS AND CONCLUSIONS

Beyond the limits of applicability of the existing thick plate theories, various orders of the expanded approximate theories of a thick plate have been applied to analyse the vibration and stability problems of a simply supported thick plate subjected to in-plane forces. In the present analysis, only the out-of-plane problems have been analysed and some useful data for the natural frequencies and buckling loads of an extremely thick plate have been obtained.

The following conclusions may be drawn from the present analysis.

(1) In order to verify the accuracy of the present results, convergence properties of the numerical solutions according to the order of approximate theories are examined. Convergence properties of the minimum natural frequencies and the buckling loads for a simply supported square plate have been examined in detail. An estimation of the approximate order of the governing equations may be concluded according to the thickness parameter. The present results obtained by $M = 5$ are considered to be accurate enough for extremely thick plates and can be regarded as the benchmark data of the problem.

(2) The minimum natural frequencies of a simply supported thick plate subjected to in-plane initial tensile and/or compressive forces have been obtained for all the thickness parameters and several displacement modes. In the cases of thick plates, the minimum natural frequencies for higher displacement modes are more affected than those for lower displacement modes by increased compressive forces.

In the case of a square plate, for comparatively thinner plates with the thickness parameter a/h larger than about 7.08, the lowest buckling load appears at the first displacement mode $r = s = 1$. However, for thicker plates with smaller values of the thickness parameter, lower buckling loads appear at higher displacement modes. When the aspect ratio becomes larger, lower buckling loads appear also at higher displacement modes.

(3) For the present range of the thickness parameter, reasonably accurate numerical solutions are obtained by $M = 2$–5. The present approximate theories can predict the natural frequencies and buckling loads of an extremely thick plate more accurately when compared to other refined theories and the classical theory. It can be said that two-dimensional higher-order plate theories in the present paper are very useful for the vibration and stability analyses of a thick plate as extended theories of the classical thin plate theory.

REFERENCES

Brunelle, E. J. and Robertson, S. R. (1974). Initially stressed Mindlin plates. *AIAA J.* **12,** 1036–1045.

Brunelle, E. J. and Robertson, S. R. (1976). Vibrations of an initially stressed thick plate. *J. Sound Vibr.* **45,** 405–416.

Doong, J.-L. (1987). Vibration and stability of an initially stressed thick plate according to a high-order deformation theory. *J. Sound Vibr.* **113,** 425–440.

Herrmann, G. H. and Armenakas, A. E. (1962). Vibration and stability of plates under initial stress. *Trans. ASCE* **127,** 458–491.

Liew, K. M., Hung, K. C. and Lim, M. K. (1993). A continuum three-dimensional vibration analysis of thick rectangular plates. *Int. J. Solids Structures* **30,** 3357–3379.

Matsunaga, H. (1986). On the analysis of displacement and stress distributions of a thick elastic plate by two-dimensional higher order theory. *Trans A.I.J.* **367,** 48–58 (in Japanese).

Matsunaga, H. (1992). The application of a two-dimensional higher-order theory for the analysis of a thick elastic plate. *Comput. Structures* **45,** 633–648.

Mindlin, R. D. (1951). Influences of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. Appl. Mech.* **18,** 31–38.

Reddy, J. N. and Phan, N. D. (1985). Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory. *J. Sound Vibr.* **98,** 157–170.

Srinivas, S., Joga Rao, C. V. and Rao, A. K. (1970). An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates. *J. Sound Vibr.* **12,** 187–199.